The Schmidt Measure as a Tool for Quantifying Multi-Particle Entanglement

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We present a measure of quantum entanglement which is capable of quantifying the degree of entanglement of a multi-partite quantum system. This measure, which is based on a generalization of the Schmidt rank of a pure state, is defined on the full state space and is shown to be an entanglement monotone, that is, it cannot increase under local quantum operations with classical communication and under mixing. For a large class of mixed states this measure of entanglement can be calculated exactly, and it provides a detailed classification of mixed states.

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Quantum entanglement is a still not fully understood feature of quantum mechanics. The nature of quantum correlations has been a central issue of long-lasting debates on the interpretation of quantum mechanics. With the emergence of quantum information theory, however, a more pragmatic approach to entanglement has become appropriate. Entanglement, in this context, is typically conceived as a resource for certain computational and communicational tasks. Terms like "entanglement distillation" – a process in which dilute entanglement is transformed into a more suitable form – already suggests that emphasis is put on the usefulness of quantum entanglement. In the spirit of this development, the conditions on any functional that is to *quantify* the degree of entanglement [1–4] have been formulated in terms of its behaviour under certain local quantum operations that can be implemented by remotely located parties.

For bi-partite systems several measures of mixed state entanglement have been proposed [1,2,5], each supplemented with a different physical interpretation. In the case of a multi-partite system [3,6–11] a single number giving the amount of entanglement is not sufficient: it has been shown that several "kinds of entanglement" have to be distinguished, and the quantification of the entanglement of such multi-partite quantum systems is essentially an open problem.

In this paper we introduce a functional – in the following referred to as Schmidt measure – that gives an account on the degree of entanglement of a multi-partite system with subsystems of arbitrary dimension. We will proceed as follows. Firstly, the measure of entanglement will be defined for pure states, employing a natural generalization of the Schmidt rank for bi-partite systems. The definition will then be generalized to the domain of mixed states, and we will show that this definition is consistent with conditions every entanglement measure has to satisfy. This entanglement measure provides the tool for a classification of the entanglement inherent in multi-particle states. Several partitions [11] of the composite systems in subsystems will be considered. Also, the connection to best separable approximations as studied in Ref. [13] will be established.

In Ref. [10] a certain class of multi-qubit states has been introduced, the so-called N-party cluster states $|\phi_N\rangle\langle\phi_N|$. It has been demonstrated that the minimal number of product terms is given by $2^{\lfloor N/2\rfloor}$, if one expands $|\phi_N\rangle$ in product states of N qubits. The need for a quantification of the entanglement of such states was the motivation for the investigations of this paper. The observation concerning the minimal number of product states is related to the findings of Ref. [9], in which such numbers have first been considered: it has been shown that there are two classes of tripartite entangled pure states of three qubits which cannot be transformed into each other with nonvanishing probability, the so-called W state [9] and the GHZ state [7] being representatives. One has three and the other one has two product states in the minimal decomposition in terms of product states. This statement is also made stronger in that it is pointed out that this minimal number of product terms can never be increased by means of invertible local operations. Building upon these observations one can define an entanglement monotone on the entire state space, containing the mixed states, of an arbitrary multi-partite system.

Consider an N-partite quantum system with parties $A_1, ..., A_N$ holding quantum systems with dimension $d_1, ..., d_N$, that is, the state space of the composite system is given by $\mathcal{S}(\mathcal{H})$, where $\mathcal{H} = \mathbb{C}^{d_1} \otimes ... \otimes \mathbb{C}^{d_N}$. Any $|\psi\rangle \in \mathcal{H}$ can be written in the form

$$|\psi\rangle = \sum_{i=1}^{R} \alpha_i |\psi_{A_1}^{(i)}\rangle \otimes ... \otimes |\psi_{A_N}^{(i)}\rangle, \qquad (1)$$

where $|\psi_{A_j}^{(i)}\rangle \in \mathbb{C}^{d_j}$, $j = 1, ..., N$, $\alpha_i \in \mathbb{C}$, $i = 1, ..., R$ with some R. Let r be the minimal number of product terms R in such a decomposition of $|\psi\rangle$. The Schmidt measure is now defined as $P(|\psi\rangle\langle\psi|) = \log_2 r$ [14]. In case of a bi-partite system with parties A_1 and A_2 the minimal number of product terms r is given by the Schmidt rank of the state. The Schmidt measure could be conceived as a generalization of the Schmidt rank to multi-partite systems in a similar way as the Schmidt number applies its concept to mixed states [15].

Once P is defined for pure states one can extend the defi-

nition to the full state space in a natural way. This is done by using a convex roof construction [1,16]: For a $\rho \in \mathcal{S}(\mathcal{H})$ let

$$P(\rho) = \min \sum_i \lambda_i P(|\psi_i\rangle\langle\psi_i|), \qquad (2)$$

where the minimum is taken over all possible convex combinations of the form $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ in terms of pure states $|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, ...$, with $0 \leq \lambda_i \leq 1$ for all i.

In the subsequent paragraph we will show that the Schmidt measure for mixed states defined in Eq. (2) qualifies for being an entanglement monotone in the sense of Ref. [3](see also Ref. [4]). That is, P is a proper measure of entanglement satisfying the following conditions:

(i) $P \geq 0$, and $P(\rho) = 0$ if ρ is fully separable [17].

(ii) P is a convex functional,

$$P(\lambda\rho_1 + (1-\lambda)\rho_2) \leq \lambda P(\rho_1) + (1-\lambda)P(\rho_2) \qquad (3)$$

for all $\lambda \in [0,1]$ and all $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$.

(iii) P is monotone under local generalized measurements: Let ρ be the initial state, and let one of the parties perform a (partly selective) local generalized measurement leading to the final states $\rho_1, ..., \rho_n$ with respective probabilities $p_1, ..., p_n$. Then $P(\rho) \geq \sum_{i=1}^n p_i P(\rho_i)$.

The physical interpretation of condition (i) is obvious. In condition (ii) part of the distinction between classical and quantum correlations is incorporated. If one mixes states in the sense that one dismisses the classical knowledge of what preparation procedure has actually been applied, the entanglement must not increase. Condition (iii) ensures that the expected entanglement does not grow when a selective generalized measurement is performed. In particular, condition (iii) leads to an invariance under local unitary operations, that is, $P(U\rho U^\dagger) = P(\rho)$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and all local unitary operators U. An implication of (ii) and (iii) together is that $P(\Lambda(\rho)) \leq P(\rho)$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and all completely positive trace-preserving maps $\Lambda : \mathcal{S}(\mathcal{H}) \longrightarrow \mathcal{S}(\mathcal{H})$ corresponding to a local quantum operation. These conditions (i) – (iii) that formalize the intuitive properties of an entanglement measure go back to Refs. [1] and [2].

Proposition. – P is an entanglement monotone fulfilling the conditions (i) – (iii).

Proof: Condition (i) follows immediately from the definition. Due to the convex roof construction P is also a convex functional: let ρ_1 and ρ_2 be states from $\mathcal{S}(\mathcal{H})$, and let $\rho_1 = \sum_j \mu_j |\phi_j\rangle\langle\phi_j|$ and $\rho_2 = \sum_k \eta_k |\varphi_k\rangle\langle\varphi_k|$ be the two decompositions for which the respective minima in Eq. (2) are attained. Then $\sum_j \lambda\mu_j |\phi_j\rangle\langle\phi_j| + \sum_k (1-\lambda)\eta_k |\varphi_k\rangle\langle\varphi_k|$ is a valid decomposition of $\rho \equiv \lambda\rho_1 + (1-\lambda)\rho_2$, but it is not necessarily the optimal one. Hence, $P(\lambda\rho_1 + (1-\lambda)\rho_2) \leq \lambda P(\rho_1) + (1-\lambda)P(\rho_2)$.

To see that P satisfies also condition (iii) we proceed in several steps. The local measurement can be assumed to be performed by party A_1. Any final state ρ_j, $j = 1, 2, ...$, in a local generalized measurement can be represented with the help of Kraus operators E_{ij}, $i, j = 1, 2, ...$, as

$$\rho_j = \frac{\sum_i E_{ij}\rho E_{ij}^\dagger}{p_j} \qquad (4)$$

with $p_j = \text{tr}[\sum_i E_{ij}\rho E_{ij}^\dagger]$, $\sum_{ij} E_{ij}^\dagger E_{ij} = I$. As the sum over i corresponds to a mixing which – due to the convexity property – reduces the Schmidt measure, it suffices to consider final states of the form $\rho_j = E_j\rho E_j^\dagger/p_j$ with $p_j = \text{tr}[E_j\rho E_j^\dagger]$. The trace-preserving property of the quantum operation reads as $\sum_j E_j^\dagger E_j = I$. For any pure state $|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H})$

$$P(E_j|\psi\rangle\langle\psi|E_j^\dagger/\text{tr}[E_j|\psi\rangle\langle\psi|E_j^\dagger]) \leq P(|\psi\rangle\langle\psi|) \quad (5)$$

for all $j = 1, 2, ...$. This can be seen as follows. Let $|\psi\rangle = \sum_{i=1}^r |\psi_{A_1}^{(i)}\rangle \otimes ... \otimes |\psi_{A_N}^{(i)}\rangle$ be the decomposition of $|\psi\rangle$ into products as in Eq. (1) with the minimal number of terms r. Then E_j is either invertible, and then $E_j|\psi\rangle$ has the same minimal number of product terms $r' = r$, or it is not invertible, such that $r' \leq r$. Moreover, if Eq. (5) holds for pure states $|\psi\rangle\langle\psi|$, it is also valid for arbitrary states $\rho \in \mathcal{S}(\mathcal{H})$: Let $\rho = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$ be the optimal decomposition of ρ belonging to the minimum in Eq. (2), then

$$\begin{aligned} P(\rho) &= \sum_k \lambda_k P(|\psi_k\rangle\langle\psi_k|) \\ &\geq \sum_k \lambda_k P(E_j|\psi_k\rangle\langle\psi_k|E_j^\dagger/\text{tr}[E_j|\psi_k\rangle\langle\psi_k|E_j^\dagger]) \\ &\geq P(E_j\rho E_j^\dagger/\text{tr}[E_j\rho E_j^\dagger]) \end{aligned} \qquad (6)$$

for all $j = 1, 2, ...$. The statement of condition (iii) follows from the fact that $\sum_j p_j P(E_j\rho E_j^\dagger/\text{tr}[E_j\rho E_j^\dagger]) \leq \sum_j p_j P(\rho) = P(\rho)$. $\qquad\square$

The Schmidt measure can indeed be used as a functional appropriately quantifying the entanglement of a given state of a N-partite quantum system. It cannot be increased on average under LOCC and it distinguishes between classical and quantum correlations [18]. The Schmidt measure is normalized, in that $P(|\psi\rangle\langle\psi|) = 1$ for all states $|\psi\rangle\langle\psi|$ of the form $|\psi\rangle = (|0_1...0_N\rangle + |1_1...1_N\rangle)/\sqrt{2}$. The functional is also (fully) additive on pure states: If the parties $A_1, ..., A_N$ share two N-partite quantum systems in the states $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$, respectively, then $P(|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|) = P(|\psi_1\rangle\langle\psi_1|) + P(|\psi_2\rangle\langle\psi_2|)$. In particular, $P(|\psi\rangle\langle\psi|^{\otimes n}) = nP(|\psi\rangle\langle\psi|)$ for n copies of the same state $|\psi\rangle\langle\psi|$. Let $\rho \in \mathcal{S}(\mathcal{H})$ be a fully separable state [17], then the stronger statement $P(\rho^{\otimes n} \otimes |\psi\rangle\langle\psi|^{\otimes n}) = nP(|\psi\rangle\langle\psi|)$ is also true [19].

Although the Schmidt measure of a mixed state is defined via a minimization over all possible realizations of the state, it can be calculated exactly for a quite large class of states.

This is mainly due to the fact that it is a coarse grained measure. All terms that appear in Eq. (2) are logarithms of natural numbers weighted with respective probabilities.

This fact allows for a detailed classification of multiparticle entangled states. The following investigations will be restricted to the multi-qubit case, meaning that $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$. Clearly, in a multi-partite setting a single number for the entanglement of the whole system is not sufficient to account for the possible entanglement structures. As in Ref. [11] we consider arbitrary partitions of the N-partite system with parties $A_1, ..., A_N$ into k parts, $k = 2, ..., N$. A division of the original system into k parts will be called a k-split. These parts are taken to be a system on their own with a certain higher dimension when P is evaluated. In notation such a division will be marked with brackets. For a, say, three-party system the 3-split $A_1 A_2 A_3$ and the three 2-splits $(A_1 A_2) A_3$, $(A_2 A_3) A_1$, and $(A_3 A_1) A_2$ are possible. In this classification the Schmidt measure will also be furnished with an index indicating the respective split. For each k-split the Schmidt measure does not increase on average in the course of a LOCC operation. This does not imply, however, that the Schmidt measure with respect to the reduced state of some of the parties must not increase on average under such operations.

Pure states. – For pure states the definition given in Eq. (1) can be easily applied. Take, e.g., as in Ref. [9] the three party W-state with state vector

$$|\mathrm{W}\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}. \qquad (7)$$

$P_{A_1 A_2 A_3}(|\mathrm{W}\rangle\langle\mathrm{W}|) = \log_2 3$, while the three party GHZ state, $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$, gets the value $P_{A_1 A_2 A_3}(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|) = 1$. Hence, in the 3-split P distinguishes the GHZ state, the W state and product states (value 0) [20]. Other splits with $k = 2$ reveal further information and give rise to the full classification. Table I shows the values of the Schmidt measure with respect to all possible splits for some pure states of a four-partite system.

TABLE I. Values of P for some four-qubit pure states: the four-party GHZ state with state vector $(|0000\rangle + |1111\rangle)/\sqrt{2}$, the generalized W state [9] corresponding to $(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)/2$, the cluster state of Ref. [10] $(|\phi_4\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2)$, and the product of two maximally entangled states of two qubits $(|\phi^+\rangle|\phi^+\rangle = (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)/2)$. The values of P associated with the remaining splits can be obtained from the permutation symmetry of the states.

	GHZ	W	$|\phi_4\rangle$	$|\phi^+\rangle|\phi^+\rangle$
$A_1 A_2 A_3 A_4$	1	2	2	2
$(A_1 A_2) A_3 A_4$	1	$\log_2 3$	1	1
$(A_1 A_2)(A_3 A_4)$	1	1	1	0
$(A_1 A_3)(A_2 A_4)$	1	1	2	2
$(A_1 A_2 A_3) A_4$	1	1	1	1

Mixed states. – In mixed quantum states both quantum entanglement and classical correlations may be present. In order to calculate the Schmidt measure of a mixed state a minimization over decompositions of the state is required. Upper bounds of P follow immediately from the definition: If $\rho = \sum \eta_i |\psi_i\rangle\langle\psi_i|$ is any not necessarily optimal decomposition of a state $\rho \in \mathcal{S}(\mathcal{H})$, then $\sum_i \eta_i P(|\psi_i\rangle\langle\psi_i|)$ is an upper bound of $P(\rho)$. For many states P can however be fully evaluated. Consider, e.g., two parties A_1 and A_2 sharing two qubits in the Werner state [12]

$$\rho_{\mathrm{W}}(\lambda) = \lambda|\psi^-\rangle\langle\psi^-| + (1-\lambda)I/4, \qquad (8)$$

with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, $0 \le \lambda \le 1$. As all pure states in the range of $\rho_{\mathrm{W}}(\lambda)$ have Schmidt measure 0 or 1, one has to identify in any decomposition $\rho_{\mathrm{W}}(\lambda) = \sum_i \eta_i |\psi_i\rangle\langle\psi_i|$ the terms with Schmidt measure 0 (product states) or 1 (entangled states). Hence, the Schmidt measure is given by $P(\rho_{\mathrm{W}}(\lambda)) = 1 - s$, where s is the weight of the separable state that can be maximally subtracted from $\rho_{\mathrm{W}}(\lambda)$ while maintaining the semi-positivity of the state, and therefore [13],

$$P(\rho_{\mathrm{W}}(\lambda)) = \begin{cases} \frac{3}{2}\lambda - \frac{1}{2}, & \text{for } 1/3 < \lambda \le 1, \\ 0, & \text{for } 0 \le \lambda \le 1/3. \end{cases} \qquad (9)$$

In other words, P is just given by the weight of the inseparable state in the best separable decomposition in the sense of Ref. [13]. As follows from the observation that the Schmidt rank of all states in the range of any state from $\mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is smaller or equal to 2, this statement holds for all mixed states of two parties holding qubits. As a corollary we find that the parameter of the best separable decomposition in 2×2 systems is an entanglement monotone.

For more than two parties several different entanglement classes can be distinguished. The Schmidt measure is again defined for all possible k-splits, $k = 2, ..., N$. If the N-partite system is separable with respect to a particular k-split, the value of the corresponding Schmidt measure is 0. Since not only condition (i) of the conditions of the entanglement monotone is satisfied, but also the stronger statement that $P(\rho) = 0$ iff ρ is fully separable, the Schmidt measure with respect to a certain split gives an account on the separability of the state taking this split. For three qubits, e.g., the classes of so-called one-qubit biseparable states, two-qubit biseparable states, three-qubit biseparable states, and fully separable states [11] can be distinguished. However, the Schmidt measure reveals more structure, since the entanglement – if present – is also quantified. As an example, consider the state

$$\begin{aligned} \rho(\lambda, \mu) = &\, \lambda|\phi^+_{A_1 A_2}\rangle|0_{A_3}\rangle\langle 0_{A_3}|\langle\phi^+_{A_1 A_2}| \\ &+ \mu|\phi^+_{A_2 A_3}\rangle|0_{A_1}\rangle\langle 0_{A_1}|\langle\phi^+_{A_2 A_3}| \\ &+ (1 - \lambda - \mu)|\phi^+_{A_3 A_1}\rangle|0_{A_2}\rangle\langle 0_{A_2}|\langle\phi^+_{A_3 A_1}|, \quad (10) \end{aligned}$$

$|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $0 \le \lambda, \mu \le 1$. For $\lambda = \mu = 1/3$ this state reduces to the three-party molecule state ρ_{M} studied in Ref. [22]. The Schmidt measure $P_{A_1 A_2 A_3}(\rho_{\mathrm{M}}) = 1$ is equal to the Schmidt measure of the state where A_1 and A_2 hold a $|\phi^+\rangle\langle\phi^+|$ state and A_3 is in the state $|0\rangle\langle 0|$, as the

mere classical ignorance of which parties are actually holding the Bell state cannot increase the amount of entanglement (compare also Ref. [23]). In Table II the values of the Schmidt measures of all splits of the states $\rho(\lambda, \mu)$, ρ_M, and $\rho_G(\lambda) = \lambda|\text{GHZ}\rangle\langle\text{GHZ}| + (1 - \lambda)|000\rangle\langle 000|, 0 \leq \lambda \leq 1$, are shown.

TABLE II. The Schmidt measure P for some mixed quantum states.

	$\rho_G(\lambda)$	ρ_M	$\rho(\lambda, \mu)$
$A_1 A_2 A_3$	λ	1	1
$(A_1 A_2) A_3$	λ	2/3	$1 - \lambda$
$(A_1 A_3) A_2$	λ	2/3	$\lambda + \mu$
$(A_2 A_3) A_1$	λ	2/3	$1 - \mu$

To summarize, we have proposed a new functional which quantifies the entanglement of quantum systems shared by N parties in possibly mixed states. It has the property to be an entanglement monotone, implying that it does increase on average under LOCC operations. Surprisingly, it turns out to be analytically computable for a large class of states, and it leads to a detailed classification of states. It is the hope that this measure of entanglement provides a helpful pragmatic tool for investigating the rich structure of multi-particle entanglement.

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[18] Among the postulates for an entanglement measure in the strict sense of Ref. [4] P does not satisfy a continuity criterion. In particular, it is not continuous in a region close to n-fold products $|\psi\rangle\langle\psi|^{\otimes n}$ of pure states. Therefore, P does not have to coincide with the von-Neumann entropy of one subsystem for a bi-partite system in a pure state.

[19] For general mixed states P is subadditive, $P(\rho \otimes \sigma) \leq P(\rho) + P(\sigma)$ for all $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, where ρ and σ are states of quantum systems held by the same parties.

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